

# Luttinger Revisited—The Renormalization Group Approach

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Luttinger's contributions abound in different parts of many-body physics. Here I review the ones that appear when one uses the Renormalization Group (RG) to study the subject: the Luttinger Liquid, Luttinger's Theorem (on the volume of the Fermi surface) and the Kohn–Luttinger Theorem on the superconducting instability of all metals as one approaches absolute zero.

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**KEY WORDS:** Renormalization group; Luttinger liquid; bosonization; Fermi liquid; BCS instability.

## 1. INTRODUCTION

As someone who entered condensed matter physics at a ripe old age, my exposure to it has been rather unconventional, guided more by my research interests than by a standard curriculum. In this process I noticed that the name Luttinger kept cropping up all over the place. Here I discuss the instances that arose in my attempts to apply the Renormalization Group (RG) to understanding many-body physics. What follows is a rather personal view of these topics, as compared to an objective review. Given the context, this seems reasonable.

It is amusing that the RG leads to many of these results since Luttinger's methods were quite different and could be summarized as follows:

- Master field theoretic methods as applied to many-body physics.
- Attack the problem frontally and try to demolish it.

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The RG approach is roughly as follows:

- Set up the problem as a multiple (path) integral.
- Keep chipping away at the integrals, saving the most difficult (singular) part for last, with no intention of doing it.

I met Luttinger only once, following a colloquium I gave on these topics. It was a brief encounter of the first kind, i.e., I did not get time to find out in depth his reaction to these alternative approaches. My own view is this. While it is very satisfying to be able to recast the results of Luttinger or Landau in RG language (which makes them easy for some of us to understand and extend), each time that happens, my respect for the original inventor of these ideas only increases, for I ask: “How did he do it without any of this machinery to help him?” I am therefore honored to contribute these personal remarks in a volume dedicated to this remarkable physicist.

## 2. THE RG APPROACH

About a decade ago, when I got into the business, high  $T_c$  materials had been discovered and Anderson had challenged the community to find an alternative to Landau’s Fermi liquid theory. In one dimension, there was a concrete case of a non-Fermi liquid, which had been dubbed the Luttinger liquid by Haldane.<sup>(1)</sup> I will return to this topic shortly, but must add that the model was posed by Luttinger<sup>(2)</sup> and correctly solved by Mattis and Lieb,<sup>(3)</sup> and is isomorphic to the massless Thirring model in quantum field theory. The question in high  $T_c$  was whether two dimensional fermions behaved more like this case or like the three dimensional case, where Fermi liquids ruled.

At that time I had already become a great fan of the RG, having seen its awesome power in the realm of critical phenomena. I wanted to apply it to the two-dimensional Fermi system to determine its fate. The basic idea was simple. First the free system in the low energy region, *which for fermions is near the Fermi surface*, would be cast in the form of a functional integral, and an RG that left its action fixed would be determined. Then, all interactions would be viewed as perturbations of this fixed point and classified as relevant, irrelevant and marginal. Only the relevant and marginal ones could possibly destabilize the Fermi liquid. I published my preliminary findings in 1991 in an issue of *Physica* devoted to Michael Fisher’s sixtieth birthday<sup>(4)</sup> (since I felt this was exactly the kind of stuff Michael would enjoy) and followed it with a very detailed RMP article in 1994.<sup>(5)</sup> I became aware that Benfatto and Gallvoti<sup>(6)</sup> and Feldman and

Trubowitz<sup>(7)</sup> had published results on this topic. Their approach was a lot more formal than I was accustomed to. I have given some additional references to subsequent work along these lines in ref. 6 (After the above mentioned work of Mattis and Lieb I had developed a healthy respect for doing things carefully and realized that sometimes the correct *physics* emerges only when one understands the mathematics, in this example, unitarily inequivalent representations, correctly. In the present case however, I am confident my approach has the right physics.) Polchinski,<sup>(8)</sup> who was interested in effective field theories for particle physics, independently arrived at the central idea in a somewhat more schematic description in 1992. Anderson had mentioned this approach in his book.<sup>(9)</sup>

I decided I would start with  $d=1$  as a warm up to see how well the method did. Let us begin with the fact that in  $d=1$ , free spinless fermions hopping on a lattice are described by the following hamiltonian in momentum spaces:

$$H_0 = \int_{-\pi}^{\pi} \frac{dK}{2\pi} \psi^\dagger(K) \psi(K) E(K) \quad (1)$$

$$E(K) = -\cos K \quad (2)$$

where

$$\{\psi(K), \psi^\dagger(K')\} = 2\pi\delta(K - K') \quad (3)$$

The Fermi sea is obtained by filling all negative energy states, i.e., those with  $|K| \leq K_F = \pi/2$ , which corresponds to half-filling, the case I will specialize on here. The Fermi surface consists of just two points  $|K| = \pm\pi/2$ . It is clear that the ground state is a perfect conductor since we can move a particle just below the Fermi surface to just above it at arbitrarily small energy cost.

Now we argue that at weak coupling, only modes near  $\pm K_F$  will be activated. Thus we will linearize the dispersion relation  $E(K) = -\cos K$  near these points and work with a cut-off  $A$ :

$$H_0 = \sum_i \int_{-A}^A \frac{dk}{2\pi} \psi_i^\dagger(k) \psi_i(k) k \quad (4)$$

where

$$k = |K| - K_F \quad (5)$$

$$i = L, R \quad (\text{left or right}) \quad (6)$$

Next we will write down a  $T=0$  partition function for the noninteracting fermions. This will be a Grassmann integral only over the degrees of freedom within a cut-off  $\Lambda$  of the Fermi surface. We will then find an RG transformation that lowers the cut-off but leaves the free-field action,  $S_0$ , invariant. With the RG well defined, we will look at the generic perturbations of this fixed point and classify them as usual. If no relevant operators show up, we will still have a scale-invariant gapless system. If, on the other hand, there are generic relevant perturbations, we will have to see to which new fixed point the system flows. (The new one could also be gapless.) The stability analysis can be done perturbatively. *In particular, if a relevant perturbation takes us away from the original fixed point, nothing at higher orders can ever bring us back to this fixed point.*

Let us then begin with the partition function for our system of fermions:

$$Z_0 = \int \prod_{i=L,R} \prod_{|k| < \Lambda} d\psi_i(\omega k) d\bar{\psi}_i(\omega k) e^{S_0} \quad (7)$$

$$S_0 = \sum_{i=L,R} \int_{-A}^A \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \bar{\psi}_i(\omega k)(i\omega - k) \psi_i(\omega k) \quad (8)$$

This is just a product of functional integrals for the Fermi oscillators at each momentum with frequency  $\Omega_0(k) = k$ .

The first step in the RG transformation is to integrate out all  $\psi(k\omega)$  and  $\bar{\psi}(k\omega)$  with

$$A/s \leq |k| \leq A \quad (9)$$

and all  $\omega$ . Thus our phase space has the shape of a rectangle, infinite in the  $\omega$  direction, but finite in the  $k$  direction. (Consult Fig. 1 for details.) This shape will be preserved under the RG transformation. Since there is no real relativistic invariance here, we will make no attempt to treat  $\omega$  and  $k$  on

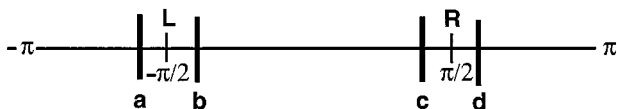


Fig. 1. The figure shows the regions of momentum space being integrated out in the  $d=1$  spinless fermion problem. The thick lines stand for the slices of width  $dA$ . They lie a distance  $A$  from the Fermi points  $L$  and  $R$ . In the ZS graph which has zero momentum transfer, both lines lie on the same slice and the  $\omega$ -integral gives zero. In the ZS' graph, the momentum transfer  $\pi$ , connects  $a$  and  $c$  (which have opposite energies) and  $b$  and  $d$  similarly related. In the BCS diagram the loop momenta are equal and opposite and correspond to  $a, d$  or  $b, c$ .

an equal footing. Allowing  $\omega$  to take all values allows us to extract an effective hamiltonian operator at any stage in the RG since locality in time is assured.

Since the integral is gaussian, the result of integrating out fast modes is just a numerical prefactor which we throw out. The surviving modes now have their momenta going from  $-A/s$  to  $A/s$ . To make this action a fixed point we define rescaled variables:

$$\begin{aligned} k' &= sk \\ \omega' &= s\omega \\ \psi'_i(k'\omega') &= s^{-3/2}\psi_i(i\omega) \end{aligned} \quad (10)$$

Ignoring a constant that comes from rewriting the measure in terms of the new fields, we see that  $S_0$  is invariant under the mode elimination and rescaling operations.

We can now consider the effect of perturbations on this fixed point. Rather than turn on the perturbation corresponding to any particular interaction (say nearest neighbor), we will perform a more general analysis. The result for the particular cases will be subsumed by this analysis.

## 2.1. Quadratic Perturbations

First consider perturbations which are quadratic in the fields. These must necessarily be of the form

$$\delta S_2 = \sum_{i=L,R} \int_{-A}^A \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \mu(k\omega) \bar{\psi}_i(\omega k) \psi_i(\omega k) \quad (11)$$

assuming symmetry between left and right fermi points.

Since this action separates into slow and fast pieces, the effect of mode elimination is simply to reduce  $A$  to  $A/s$  in the integral above. Rescaling moments and fields, we find

$$\mu'(\omega', k', i) = s\mu(\omega, k, i) \quad (12)$$

We get this factor  $s$  as a result of combining a factor  $s^{-2}$  from rewriting the old momenta and frequencies in terms of the new and a factor  $s^3$  which comes from rewriting the old fields in terms of the new.

Let us expand  $\mu$  in a Taylor series

$$\mu(k, \omega) = \mu_{00} + \mu_{10}k + \mu_{01}i\omega + \dots + \mu_{nm}k^n(i\omega)^m + \dots \quad (13)$$

The constant piece is a relevant perturbation:

$$\mu_{00} \rightarrow s\mu_{00} \quad (14)$$

This relevant flow reflects the readjustment of the Fermi sea to a change in chemical potential. As for the next two terms, they are marginal. When we consider quartic interactions, it will be seen that mode elimination will produce relevant and marginal terms of the above form even if they were not there to begin with just as  $\phi^4$  theory. The way to deal with all relevant quadratic terms are discussed in Ref. [5]. The marginal terms will modify the Fermi velocity and rescale the field. As for higher order terms in Eq. (13), they are irrelevant under the RG mentioned above.

Note that in order to define the RG, we need to know the location of the Fermi momentum  $K_F$ , since we zero-in on the Fermi surface (a pair of points in this case) as the RG acts. Here again we find a very useful result due to Luttinger: the Fermi momentum is determined by the number density and the relation between them is the same as in the free case. This result can be exploited here if we work with fixed number of particles rather than a fixed chemical potential. This means that we must determine the requisite  $\mu$  perturbatively as we go along. In my long article I spell out the details of this process, but the main point is that in the RG language, this is the way to fine-tune (the coefficient of) a relevant operator to zero. This operator does not produce a gap, but instead moves the Fermi surface unless we kill it. In the work of Luttinger,<sup>(10)</sup> Kohn and Luttinger,<sup>(11)</sup> Luttinger and Ward<sup>(12)</sup> this process ensures a nice perturbation series.

## 2.2. Quartic Perturbations: The RG at Tree Level

We now turn on the quartic interaction whose most general form is

$$\delta S_4 = \frac{1}{2! 2!} \int_{K\omega} \bar{\psi}(4) \bar{\psi}(3) \psi(2) \psi(1) u(4, 3, 2, 1) \quad (15)$$

where

$$\bar{\psi}(i) = \bar{\psi}(K_i, \omega_i) \text{ etc.} \quad (16)$$

$$\int_{K\omega} = \left[ \prod_{i=1}^4 \int_{-\pi}^{\pi} \frac{dK_i}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega_i}{2\pi} \right] \times [2\pi\bar{\delta}(K_1 + K_2 - K_3 - K_4) 2\pi\delta(\omega_1 + \omega_2 - \omega_3 - \omega_4)] \quad (17)$$

and  $\bar{\delta}$  enforces momentum conservation mod  $2\pi$ , as is appropriate to any lattice problem. A process where lattice momentum is violated in multiples

of  $2\pi$  is called an *umklapp* process. The delta function containing frequencies enforces time translation invariance. The coupling function  $u$  is anti-symmetric under the exchange of its first or last two arguments among themselves since that is true of the Grassmann fields that it multiplies. Thus the coupling  $u$  has all the symmetries of the full vertex function  $\Gamma$  with four external lines.

Let us now return to the general interaction, Eqs. (15)–(17), and restrict the momenta to lie within  $\Lambda$  of either Fermi point  $L$  or  $R$ . Using a notation where  $L$  (left Fermi point) and  $R$  (right Fermi point) become discrete a label  $i=L$  or  $R$  and 1–4 label the frequencies and momenta (measured from the appropriate Fermi points). Eqs. (15)–(17) become

$$\delta S_4 = \frac{1}{2! 2!} \sum_{i_1 i_2 i_3 i_4 = L, R} \int_{K\omega}^{\Lambda} \bar{\psi}_{i_4}(4) \bar{\psi}_{i_3}(3) \psi_{i_2}(2) \psi_{i_1}(1) u_{i_4 i_3 i_2 i_1}(4, 3, 2, 1) \quad (18)$$

where

$$\int_{K\omega}^{\Lambda} = \left[ \int_{-\Lambda}^{\Lambda} \frac{dk_1 \cdots dk_4}{(2\pi)^4} \int_{-\infty}^{\infty} \frac{d\omega_1 \cdots d\omega_4}{(2\pi)^4} \right] [2\pi\delta(\omega_1 + \omega_2 - \omega_3 - \omega_4)] \\ \times [2\pi\delta(\varepsilon_{i_1}(K_F + k_1) + \varepsilon_{i_2}(K_F + k_2) - \varepsilon_{i_3}(K_F + k_3) - \varepsilon_{i_4}(K_F + k_4))] \quad (19)$$

and

$$\varepsilon_i = \pm 1 \quad \text{for } R, L \quad (20)$$

Let us now implement the RG transformation with this interaction. This proceeds exactly as in  $\phi^4$  theory. Let us recall how it goes. If schematically

$$Z = \int d\phi_{<} d\phi_{>} e^{-\phi_{<}^2 - \phi_{>}^2} e^{-u(\phi_{<} + \phi_{>})^4} \quad (21)$$

is the partition function and we are eliminating  $\phi_{>}$ , the fast modes, the effective  $u$  for  $\phi_{<}$  has two origins. First, we have a term  $-u\phi_{<}^4$  which is there to begin with, called the *tree level* term. Next, there are terms generated by the  $\phi_{>}$  integration. These are computed in a cumulant expansion and are given by Feynman diagrams whose internal momenta lie in the range being eliminated. The loops that contribute to the flow of  $u$  begin at order  $u^2$ .

Let us first do the order  $u$  tree level calculation for the renormalization of the quartic interaction. This gives us just Eq. (19) with  $\Lambda \rightarrow \Lambda/s$ . If we

now rewrite this in terms of new momenta and fields, we get an interaction with the same kinematical limits as before and we can meaningfully read off the coefficient of the quartic-Fermi operators as the new coupling function. We find

$$u'_{i_4 i_3 i_2 i_1}(k'_i, \omega'_i) = u_{i_4 i_3 i_2 i_1}(k'_i/s, \omega'_i/s) \quad (22)$$

The reader who carries out the intermediate manipulations will notice an important fact:  $K_F$  never enters any of the  $\delta$  functions: either all  $K_F$ 's cancel in the nonumklapp cases, or get swallowed up in multiples of  $2\pi$  (in inverse lattice units) in the umklapp cases due to the periodicity of the  $\bar{\delta}$ -function. As a result the momentum  $\delta$  functions are free of  $K_F$  and scale very nicely under the RG transformation:

$$\bar{\delta}(k) \rightarrow \bar{\delta}(k'/s) \quad (23)$$

$$= s\bar{\delta}(k') \quad (24)$$

Turning now to Eq. (22), if we expand  $u$  in a Taylor series in its arguments and compare coefficients, we find readily that the constant term  $u_0$  is marginal and the higher coefficients are irrelevant. Thus  $u$  depends only on its discrete labels and we can limit the problem to just a few coupling constants instead of the coupling function we started with. Furthermore, all reduce to just one coupling:

$$u_0 = u_{LRLR} = u_{RLRL} = -u_{RLLR} = -u_{LRRR} \quad (25)$$

Other couplings corresponding to  $LL \rightarrow RR$  are wiped out by the Pauli principle since they have no momentum dependence and can't have the desired antisymmetry.

The tree level analysis readily extends to couplings with six or more fields. All these are irrelevant, even if we limit ourselves to constant ( $\omega$  and  $k$  independent) couplings.

In summary, the RG tells us that the most general low energy theory for spinless fermions in  $d=1$  has, at tree level, a single marginal coupling constant.

To determine the ultimate fate of the coupling  $u_0$ , marginal at tree level, we must turn to the one loop RG effects.

### 2.3. RG at One Loop: The Luttinger Liquid

Let us begin with the action with the quartic interaction and do a mode elimination. Consult Fig. 1 for details. To order  $u$ , this leads to an induced quadratic term represented by the tadpole graph in Fig. 2. We set



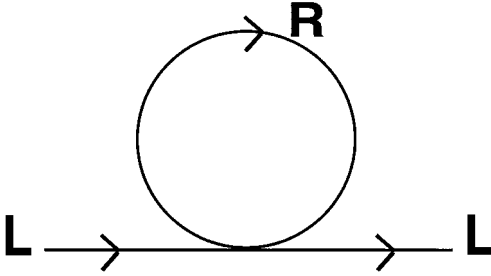


Fig. 2. The tadpole graph which renormalizes the fermion at one loop. It has no dependence on  $k$ , the deviation of the external momentum from  $K_F$  or  $\omega$ . We have used this freedom to set both these to zero on the external legs. The effect of this graph may be neutralized by a counter-term corresponding to a change in chemical potential. One may do this if one wants to preserve  $K_F$ .

$\omega = k = 0$  for the external legs (since the dependence on these is irrelevant) and have chosen them to lie at  $L$ , the left Fermi point. The integral given by the diagram produces a momentum independent term of the form  $\delta\mu \bar{\psi}_L \psi_L$ . But we began with no such term. Thus we do not have a fixed point in this case. Instead we must begin with some term  $\delta\mu^* \bar{\psi}_L \psi_L$  such that upon renormalization it reproduces itself. We find it by demanding that

$$\delta\mu^* = s \left[ \delta\mu^* - u_0^* \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{A/s < |k| < A} \frac{dk}{2\pi} e^{i\omega 0^+} \frac{1}{i\omega - k} \right] \quad (26)$$

where we have used the zeroth order propagator and the fact that to this order any  $u_0 = u_0^*$ . The exponential convergence factor is the one always introduced to get the right answer for, say, the ground state particle density using  $\langle \bar{\psi}\psi \rangle$ . Doing the  $\omega$  integral, we get

$$\delta\mu^* = s \left[ \delta\mu^* - u_0^* \int_{A/s < |k| < A} \frac{dk}{2\pi} \theta(-k) \right] \quad (27)$$

$$= s \left[ \delta\mu^* - \frac{\Lambda u_0^*}{2\pi} (1 - 1/s) \right] \quad (28)$$

It is evident that the fixed point is given by

$$\delta\mu^* = \frac{\Lambda u_0^*}{2\pi} \quad (29)$$

Alternatively, we could just as well begin with the following relation for the renormalized coupling

$$\delta\mu' = s \left[ \delta\mu - u_0^* \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{\Lambda/s < |k| < \Lambda} \frac{dk}{2\pi} e^{i\omega 0^+} \frac{1}{i\omega - k} \right] \quad (30)$$

which implies the flow

$$\frac{d\mu}{dt} = \mu - \frac{u_0^*}{2\pi} \quad (31)$$

assuming we choose to measure  $\mu$  in units of  $\Lambda$ . The fixed point of this equation reproduces Eq. (29).

We can find  $\delta\mu^*$  in yet another way with no reference to the RG. If we calculate the inverse propagator in the cut-off theory to order  $u$ , we will find

$$G^{-1} = i\omega - k - \frac{\Lambda u_0}{2\pi} \quad (32)$$

indicating that the Fermi point is no longer given by  $k = 0$ . To reinstate the old  $K_F$  as interactions are turned on, we must move the chemical potential away from zero and to the value  $\delta\mu = \Lambda u_0 / 2\pi$ . Thus the correct action that gives us the desired  $K_F$ , for this coupling, to this order, is then schematically given by

$$S = \bar{\psi}(i\omega - k)\psi + \frac{\Lambda u_0}{2\pi} \bar{\psi}\psi + \frac{u_0}{2! 2!} \bar{\psi}\psi\bar{\psi}\psi \quad (33)$$

An RG transformation on this action would not generate the tadpole graph contribution.

*A very important point which will appear again is this: we must fine tune the chemical potential as a function of  $u$ , not to maintain criticality (as one does in  $\phi^4$  where the bare mass is varied with the interaction to keep the system massless) but to retain the same particle density.* (To be precise, we are keeping fixed  $K_F$ , the momentum at which the one-particle Greens function has its singularity. This amounts to keeping the density fixed, following Luttinger (1960).) If we kept  $\mu$  at the old value of zero, the system would flow away from the fixed point with  $K_F = \pi/2$ , not to a state with a gap, but to another gapless one with a smaller value of  $K_F$ . This simply corresponds to the fact if the total energy of the lowest energy particle that can be added to the system, namely  $\mu$ , is to equal 0, the kinetic

energy at the Fermi surface must be slightly negative so that the repulsive potential energy with the others in the sea brings the total to zero.

Let us now turn our attention to the order  $u_0^2$  graphs that renormalize  $u_0$ . These are shown in Fig. 3. The increment in  $u_0$ , hereafter simply called  $u$ , is given by the sum of the ZS (zero-sound), ZS' and BCS graphs. The analytical formula for the increment in  $u$  is

$$\begin{aligned} du(4321) = & \int u(6351) u(4526) G(5) G(6) \delta(3+6-1-5) d5 d6 \\ & - \int u(6451) u(3526) G(5) G(6) \delta(6+4-1-5) d5 d6 \\ & - \frac{1}{2} \int u(6521) u(4365) G(5) G(6) \delta(5+6-1-2) d5 d6 \end{aligned} \quad (34)$$

where 1 to 4 stand for all the attributes of the (slow) external lines, 5 and 6 stand for all the attributes of the two (fast) internal lines: momenta (restricted to be within the region being eliminated), and frequencies;  $G$  are the propagators and the  $\delta$  functions are for ensuring the conservation of momenta and frequencies and  $\int d5 d6$  stands for sums and integrals over the attributes 5 and 6. (In the figure the momenta 1 to 6 have been assigned some special values (such as  $5 = K$  in Fig. 3a) that are appropriate to the problem at hand. The formula is very general as it stands and describes other situations as well.) The couplings  $u$  are functions of all these attributes, with all the requisite antisymmetry properties. (The order in which the legs are labeled in a is important due to all the minus signs. The above equations have been written to hold with the indicated order of arguments. In their present form they are ready to be used by a reader who wants to include spin.)

This is the master formula we will invoke often. It holds even in higher dimensions, if we suitably modify the integration region for the momenta.

Readers familiar with Feynman diagrams may obtain this formula by drawing all the diagrams to this order in the usual Feynman graph expansion, but allowing the loop momenta to range only over the modes being eliminated. In the present case, these are given by the four thick lines labeled a, b, c and d in Fig. 1 where each line stands for a region of width  $dA$  located at the cut-off, i.e., a distance  $A$  from the Fermi points. The external momenta are chosen to be  $(4321) = (LRLR)$ , at the Fermi surface. All the external  $k$ 's and  $\omega$ 's are set equal to zero since the marginal coupling  $u$  has no dependence on these. This has two consequences. First, the loop frequencies in the ZS and ZS' graphs are equal, while those in the

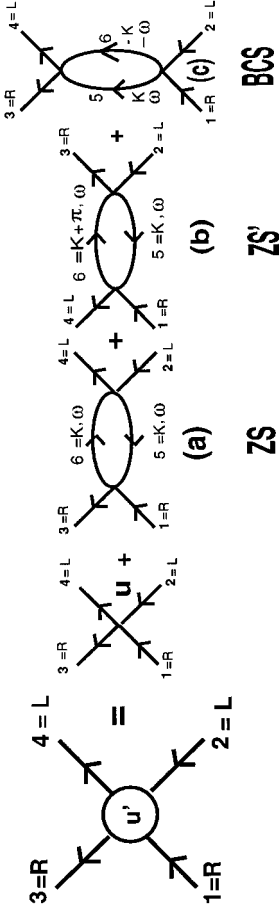


Fig. 3. The one loop graphs for  $\beta(u)$  for the spinless fermions in  $d = 1$ . The loop momenta lie in the shell of width  $dA$  being eliminated. The external frequencies being all zero, the loop frequencies are equal for ZS (Fig. 3a) and ZS' (Fig. 3b) graphs and equal and opposite for the BCS graph (Fig. 3c). The ZS graph does not contribute since both loop momenta are equal (the momentum transfer  $Q$  at the left vertex is 0) and lie a distance  $A$  above or below the Fermi surface and the  $\omega$  integration vanishes when the poles lie on the same half-plane. The ZS' graph has momentum transfer  $\pi$  at the left and right ends. This changes the sign of the energy of the line entering left the vertex. The  $\omega$  integral is nonzero, the poles being on opposite half-planes. The BCS graph (c) also survives since the loop momenta are equal and opposite (since the incoming momentum is zero) and this again makes the poles go to the opposite half-planes because the lines have *opposite* frequencies. The labels 1...6 refer to the master Eq. (34).

BCS graph are equal and opposite. (The labels Zero sound and BCS describe the topology of these graphs and not literally these phenomena.) Second, the momentum transfers at the left vertex are  $Q = K_1 - K_3 = 0$  in the ZS graph,  $Q' = K_1 - K_4 = \pi$  in the ZS' graph, while the total momentum in the BCS graph is  $P = K_1 + K_2 = 0$ . Therefore if one loop momentum  $5 = K$  lies in any of the four shells in Fig. 1, so does the other loop momentum  $6$  which equals  $K$ ,  $K + \pi$  or  $-K$  in the ZS, ZS' and BCS graphs respectively. Thus we may safely eliminate the momentum conserving  $\delta$  function in Eq. (34) using  $\int d6$ . This fact, coupled with

$$E(-K) = E(K) \quad (35)$$

$$E(K' = K \pm \pi) = -E(K) \quad (36)$$

leads to

$$\begin{aligned} du(LRLR) &= \int_{-\infty}^{\infty} \int_{dA} \frac{d\omega dK}{4\pi^2} \frac{u(KRKR) u(LKLK)}{(i\omega - E(K))(i\omega - E(K))} \\ &\quad - \int_{-\infty}^{\infty} \int_{dA} \frac{d\omega dK}{4\pi^2} \frac{u(K'LKR) u(RKLK')}{(i\omega - E(K))(i\omega + E(K))} \\ &\quad - \frac{1}{2} \int_{-\infty}^{\infty} \int_{dA} \frac{d\omega dK}{4\pi^2} \frac{u(-KKLR) u(LR - KK)}{(i\omega - E(K))(-i\omega - E(K))} \end{aligned} \quad (37)$$

$$\equiv \text{ZS} + \text{ZS}' + \text{BCS} \quad (38)$$

where  $\int_{dA}$  means the momentum must lie in one of the four slices in Fig. 1.

The reader is reminded once again that the names ZS, ZS' or BCS refer only to the topologies of the graphs. To underscore this point, especially for readers who have seen a similar integral in zero sound calculations, we will now discuss the ZS graph. In the present problem the loop momentum  $K$  lies within a sliver  $dA$  of the cut-off. Both propagators have poles at the point  $\omega = -iE(k = \pm A)$ . No matter which half-plane this lies in, we can close the contour the other way and the  $\omega$  integral vanishes. This would be the case even if a small external momentum transfer ( $Q = K_3 - K_1 \ll A$ ) takes place at the left vertex since both poles would still be on the same side. This is very different from what happens in zero sound calculations where the loop momenta roamed freely within the cut-off, and in particular, go to the Fermi surface. In that case, the integral becomes very sensitive to how the external momentum transfer  $Q = K_3 - K_1$  and frequency transfer  $\Omega = \omega_3 - \omega_1$  are taken to zero since any nonzero  $Q$ ,

however small, will split the poles and make them lie on different half planes for  $k < Q$  and the integral will be nonzero. It is readily seen that

$$\int_{-\infty}^{\infty} \int_{-A}^A \frac{d\omega dk}{4\pi^2} \frac{1}{(i\omega - k)(i\omega - k - Q + i\Omega)}$$

$$= \int_{-A}^A \frac{dk}{2\pi} \frac{i}{\Omega + iq} (\theta(k) - \theta(k + q)) \quad (39)$$

where the step function  $\theta(k)$  is simply related to the Fermi function:  $f(k) = 1 - \theta(k)$ . If we keep  $\Omega \neq 0$  and send  $Q$  to zero we get zero. On the other hand if we set  $\Omega = 0$  and let  $Q$  approach zero we get (minus) the derivative of the (Fermi)  $\theta$  function, i.e., a  $\delta$ -function at the Fermi surface. Thus reader used to zero-sound physics should not be disturbed by the fact that the ZS graph makes no contribution since the connotation here is different.

Now for the ZS' graph, Fig. 3b, Eq. (37). We see that  $K$  must lie near  $L$  since  $1 = R$  and there is no  $RR$  scattering. As far as the coupling at the left vertex is concerned, we may set  $K = L$  since the marginal coupling has no  $k$  dependence. Thus  $K + \pi = R$  and the vertex becomes  $u(RLLR) = -u$ . So does the coupling at the other vertex. Doing the  $\omega$  integral (which is now nonzero since the poles are always on opposite half-planes) we obtain, upon using the fact that there are two shells (a and b in Fig. 1) near  $L$  and that  $|E(K)| = |k| = |A|$ ,

$$ZS' = u^2 \int_{dA \in L} \frac{dK}{4\pi |E(K)|} = \frac{u^2}{2\pi} \frac{d|A|}{A} \quad (40)$$

The reader may wish to check that the ZS' graph will make the same contribution to the  $\beta$ -function in the field theory approach.

The BCS graph (Eq. (37), Fig. 3c) gives a nonzero contribution since the propagators have opposite frequencies, opposite momenta, but equal energies due to time-reversal invariance  $E(K) = E(-K)$ . We notice that the factor of  $\frac{1}{2}$  is offset by the fact that  $K$  can now lie in any of the four regions a, b, c, or d. We obtain a contribution of the same magnitude but opposite sign as ZS' so that

$$du = \left( \frac{u^2}{2\pi} - \frac{u^2}{2\pi} \right) \underbrace{\frac{d|A|}{A}}_{dt} \quad (41)$$

$$\frac{du}{dt} = \beta(u) = 0 \quad (42)$$

Thus we find that  $u$  is still marginal. The flow to one loop for  $\mu$  and  $u$  is

$$\frac{d\mu}{dt} = \mu - \frac{u}{2\pi} \quad (43)$$

$$\frac{du}{dt} = 0 \quad (44)$$

There is a line of fixed points:

$$\mu = \frac{u^*}{2\pi} \quad (45)$$

$$u^* = \text{arbitrary} \quad (46)$$

Notice that  $\beta$  vanishes due to a cancellation between two diagrams, each of which by itself would have led to the CDW or BCS instability. When one does a mean-field calculation for CDW, one focuses on just the  $ZS'$  diagram and ignores the BCS diagram. This amounts to taking

$$\frac{du}{dt} = \frac{u^2}{2\pi} \quad (47)$$

which, if correct, would imply that any positive  $u$  grows under renormalization. If this growth continues we expect a CDW. On the other hand, if just the BCS diagram is kept we will conclude a run-off for negative couplings leading to a state with  $\langle \psi_R \psi_L \rangle \neq 0$ .

What the  $\beta$  function does is to treat these competing instabilities simultaneously and predict a scale-invariant theory.

Is this the correct prediction for the spinless model? If we consider a nearest-neighbor interaction of strength  $u$ , the exact solution of Yang and Yang (1976)<sup>(13)</sup> tells us there is no gap till  $u$  is of order unity. If the RG analysis were extended to higher loops we would keep getting  $\beta = 0$  to all orders. This follows from the Ward identity in the cut-off continuum model (Di Castro and Metzner<sup>(14)</sup>) which reflects the fact that in this model, the number of fermions of type  $L$  and  $R$  are separately conserved. The vanishing beta function also agrees with the original finding of Solyom.<sup>(15)</sup>

This model also coincides with the massless Thirring model, which is a Lorentz invariant theory with a current-current interaction. The reason is that using  $L$  or  $R$  fields and their adjoints, there is just one possible quartic interaction. As for Lorentz invariance, it was assured when we linearized the spectrum near the Fermi points.

This scale-invariant system is called the Luttinger liquid. The system has a Fermi surface at which the occupation number has a kink in slope, but no jump in value. The Fermionic Green's functions fall with anomalous powers that vary with the coupling  $u$ . The quasi-particle is totally gone, no matter how small the interaction. Under bosonization, the model maps into a free bosons. The complicated fermionic behavior is encoded in the expressions for the fermions operators in the bosonic language. If spin is included, we get two bosons, moving at different velocities, a phenomenon called spin-charge separation.

How do we ever reproduce the eventual charge density wave instability known to exist in the exact solution of the model with nearest neighbor interactions? The answer is as follows. As we move along the line of fixed points, labeled by  $u$ , the dimension of various operators will change from the free-field values. Ultimately the umklapp coupling, ( $RR \leftrightarrow LL$ ), which was suppressed by a factor  $(k_1 - k_2)(k_3 - k_4)$ , will become marginal and then relevant, as shown by Haldane.<sup>(1)</sup> If we were not at half-filling such a term would be ruled out by momentum conservation and the scale invariant Luttinger liquid would persist for all  $u$ . While this liquid provides us with an example of where the RG does better than mean-field theory, it is rather special and seems to occur in  $d = 1$  systems where the two Fermi points satisfy the conditions for *both* CDW and BCS instabilities. In higher dimensions one finds that any instability due to a divergent susceptibility is never precisely cancelled by another.

### 3. HIGHER DIMENSIONS

The extension of these methods to higher dimensions is discussed in the RMP article and this discussion will be limited to the part that makes contact with Luttinger's work.

In  $d = 2$  dimensions, I considered Fermi surfaces of arbitrary shape. In the circular case, to which I limit myself here, I found that if one took an annulus of thickness  $2A$  concentric with the Fermi circle, and let  $A \rightarrow 0$ , RG yielded two marginal *coupling functions*,  $F(\theta)$  and  $V(\theta)$ , see Fig. 4. (As in  $d = 1$ , the chemical potential had to be fine tuned to keep  $K_F$  fixed.) The function  $F$  described forward scattering and was shown to be Landau's  $F$  function. It remained marginal to all orders in the loop expansion. (I pointed out that  $A/K_F$  played the role of the small parameter  $1/N$  which made such statements possible.) The function  $V(\theta)$ , which described scattering between Cooper pairs, evolved as follows:

$$\frac{dV_m}{dt} = -c_m V_m^2 \quad (48)$$



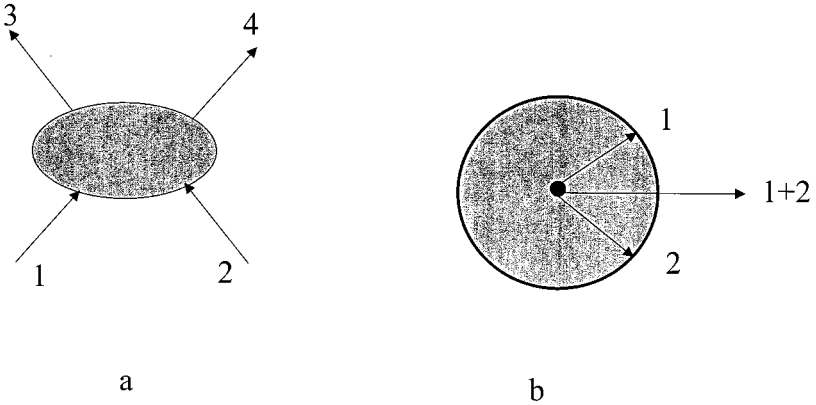


Fig. 4. (a) The quartic coupling. A label like 1 stands for three things: an angle  $\theta_1$  on the Fermi surface, a frequency  $\omega_1$  and a momentum  $K - K_F \equiv k$ , both equal to 0 since the dependence on these two is irrelevant. (b) The low energy region that survives under RG in  $d=2$ . The bandwidth  $2\Lambda$  has become as small as the thickness of the circular line. Note that if the two incoming momenta lie on this circle, the outgoing momenta must equal them:  $\theta_1 = \theta_3$  and  $\theta_2 = \theta_4$  (or the exchanged version). The angle  $\theta_1 - \theta_2 = \theta$  is the argument of Landau's  $F$  function. An exception arises if  $\theta_1 = -\theta_2$ , in which case  $\theta_3 = -\theta_4$  and the angle between these two opposing lines is the argument of the Cooper amplitude  $V$ .

where  $V_m$  was the  $m$ th Fourier coefficient (angular momentum  $m$  channel of Cooper pairs). This meant that any positive  $V_m$  flowed to zero logarithmically (an old result of Morel and Anderson<sup>(16)</sup>) while *any attractive*  $V_m$  grew in strength leading to the BCS instability. In other words, the Fermi liquid had an infinite number of unstable directions, and attraction in any angular momentum channel between Cooper pairs spelled its end.

We relate the last statement with the result of Kohn–Luttinger<sup>(17)</sup> that any Fermi system will end up superconducting as follows. The couplings  $V_m$  are not the bare couplings, say of a Hubbard or continuum model. They are the result of integrating out all modes outside the cut-off, say, using in perturbation theory. The Kohn–Luttinger result amounts to the statement that if one does this, some  $V_m$  or other will surely be negative. To see this, one need look at only the diagrams considered by Kohn–Luttinger. The fact that the loop momenta lie outside the cut-off in the RG approach do not change the conclusion that an attractive coupling will be generated from repulsive ones, as shown in my review.

#### 4. CONCLUSIONS

I have focused here on the results of Luttinger's work that arose naturally when I used the RG as a tool to study many-body fermionic

systems. I discussed a special  $d=1$  system in which superconductivity and CDW instability duke it out and neither wins, leaving behind a scale invariant system, the Luttinger liquid. It is not possible to get such a system in  $d>1$  without invoking more singular interactions than the coulomb or strong coupling. I showed that in order to zero-in on the low energy sector for fermions, namely the Fermi surface, one needs to know where it lies and here the results of Luttinger, Kohn and Ward reappear. Finally I showed that the Kohn–Luttinger result, of the inevitable superconducting instability, appears here as the statement that by the time we eliminate modes outside the cut-off and get to the physics near the Fermi surface, some Cooper coupling will surely become negative, at which point the RG takes over and predicts it will grow.

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